Problem Session 5 02/16/2018

(1) A conducting sphere of radius a carrying charges Q is surrounded by a dielectric shell of thickness $b-a$. What is the $\vec{E}$ field everywhere? Find the induced bound charges at the interfaces.

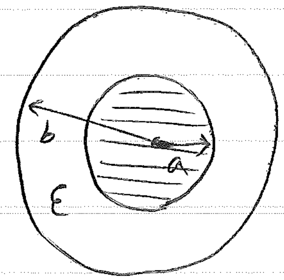
(2) Problem 5.1, Jackson.

(3) Problem 5.13, Jackson.

(1) Due to symmetry, we have:

$$\vec{E} = E(r)\,\hat{r}$$

$$\vec{D} = D(r)\,\hat{r}$$



The Gauss's law results in:

$$\oint_S \vec{D}\cdot d\vec{a} = Q_{free} \Rightarrow 4\pi r^2 D(r) = \begin{cases} 0 & r<a \\ Q & r \geq a \end{cases}$$

Thus:

$$\boxed{D(r) = \begin{cases} 0 & r<a \\ \dfrac{Q}{4\pi r^2} & r \geq a \end{cases}} \Rightarrow \boxed{E(r) = \begin{cases} 0 & r<a \\ \dfrac{Q}{4\pi \varepsilon r^2} & a \leq r < b \\ \dfrac{Q}{4\pi \varepsilon_0 r^2} & r > b \end{cases}}$$

Since $\vec{P} \propto \vec{E}$ and $\vec{\nabla}\cdot\vec{E} = 0$, we find $\boxed{\rho_{bound} = -\vec{\nabla}\cdot\vec{P} = 0}$. However:

$$\sigma_{bound} = P_{1,n} - P_{2,n} \qquad (\vec{P} = \vec{D} - \varepsilon_0 \vec{E})$$

Therefore:

$$\boxed{\sigma_{bound} = \begin{cases} \left(\dfrac{\varepsilon_0 - \varepsilon}{\varepsilon}\right) \dfrac{Q}{4\pi a^2} & r = a \\ \left(\dfrac{\varepsilon - \varepsilon_0}{\varepsilon}\right) \dfrac{Q}{4\pi b^2} & r = b \end{cases}}$$

(2) We have:

$$\vec{B} = \frac{\mu_0}{4\pi} I \oint_C \vec{dl'} \times \frac{(\vec{x}-\vec{x'})}{|\vec{x}-\vec{x'}|^3} = \frac{\mu_0}{4\pi} I \oint_C dl' \times \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x'}|}$$

But, from Stoke's theorem:

$$\oint_C \vec{dl'} \cdot \vec{A} = \int_S (\vec{\nabla}' \times \vec{A}) \cdot \hat{n}' \, da'$$

Let us write $\vec{A} = \vec{V} \times \vec{e}$, where $\vec{e}$ is a constant vector. Then:

$$\oint_C \vec{dl'} \cdot (\vec{V} \times \vec{e}) = \int_S \vec{\nabla}' \times (\vec{V} \times \vec{e}) \cdot \hat{n}' \, da' = \int_S [(\vec{e} \cdot \vec{\nabla}') V -$$

$$\vec{e} (\vec{\nabla}' \cdot V)] \cdot \hat{n}' \, da'$$

In our case, $\vec{V} = \vec{\nabla}' \left(\frac{1}{|\vec{x}-\vec{x'}|}\right)$, which implies that $\vec{\nabla}' \cdot \vec{V} = 0$ for $\vec{x} \neq \vec{x'}$. We also note that:

$$(\vec{e} \cdot \vec{\nabla}') V = (\vec{e} \cdot \vec{\nabla}') \vec{\nabla}' \left(\frac{1}{|\vec{x}-\vec{x'}|}\right) = (\vec{e} \cdot \vec{\nabla}) \vec{\nabla} \left(\frac{1}{|\vec{x}-\vec{x'}|}\right)$$

since $\vec{\nabla}' \left(\frac{1}{|\vec{x}-\vec{x'}|}\right) = -\vec{\nabla} \left(\frac{1}{|\vec{x}-\vec{x'}|}\right)$

Hence:

$$\vec{e} \cdot \oint_C \vec{dl'} \times \vec{\nabla}' \left(\frac{1}{|\vec{x}-\vec{x'}|}\right) = (\vec{e} \cdot \vec{\nabla}) \int_S \left(\vec{\nabla} \left(\frac{1}{|\vec{x}-\vec{x'}|}\right) \cdot \hat{n}'\right) da'$$

Since this relation holds for any constant vector $\vec{e}$, we must have;

$$\oint_C d\vec{\ell}' \times \vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \vec{\nabla}\int_S \left(\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\cdot\hat{n}'\right)da' = \vec{\nabla}\int \frac{-(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\cdot\hat{n}'\,da'$$

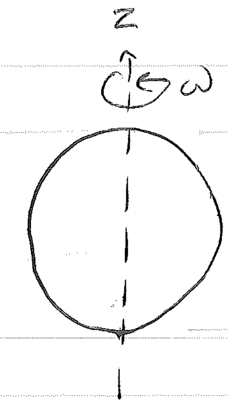
However;

$$\int_S -\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\cdot\hat{n}'\,da' = \int d\Omega' = \Omega \leftarrow$$ solid angle subtended by the loop at P

This results in:

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} I \vec{\nabla}\Omega}$$

(3) The surface current density is:



$\vec{\omega} = \omega\hat{z}$

$\vec{K} = \sigma\vec{v} = \sigma\vec{\omega}\times\vec{x} = \sigma\omega(x\hat{y} - y\hat{x})$

$= \sigma\omega a\sin\theta(\cos\phi\,\hat{y} - \sin\phi\,\hat{x})$

We have:

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{K}(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|} = \frac{\mu_0\sigma\omega a}{4\pi}\int\frac{\sin\theta'(\cos\phi'\,\hat{y} - \sin\phi'\,\hat{x})}{|\vec{x}-\vec{x}'|}\,a^2\,d\Omega'$$

Both of $\sin\theta'\cos\phi'$ and $\sin\theta'\sin\phi'$ can be expressed in terms of $Y_{1,1}(\theta',\phi')$ and $Y_{1,-1}(\theta',\phi')$. Then, from the orthogonality of Y_{lm}'s, we can see that only the $l=1$, $m=\pm1$ terms in the multipole expansion of $\frac{1}{|\vec{x}-\vec{x}'|}$ are relevant. Thus:

$$\vec{A}(\vec{x}) = \frac{\mu_0\sigma\omega a}{4\pi}\sum_{m=-1,+1}\frac{r_<}{r_>^2}\frac{4\pi}{3}\Big[Y_{1m}(\theta,\phi)\int Y^*_{1m}(\theta',\phi')\sin\theta'\cos\phi'\,d\Omega'\,\hat{y} - Y_{1m}(\theta,\phi)\int Y^*_{1m}(\theta',\phi')\sin\theta\sin\phi'\,d\Omega'\,\hat{x}\Big]$$

Note that:

$\sin\theta'\cos\phi' = -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left[Y_{1,1}(\theta',\phi') - Y_{1,-1}(\theta',\phi')\right]$, $\sin\theta'\sin\phi' = \frac{-1}{2i}\sqrt{\frac{8\pi}{3}}$

$$\left[Y_{1,1}(\theta',\phi') + Y_{1,-1}^{*}(\theta',\phi')\right]$$

This leads to:

$$\vec{A}(\vec{x}) = \frac{\mu_0 \sigma \omega a}{3}\frac{r_<}{r_>^2}\left[-\sqrt{\frac{2\pi}{3}}\left(Y_{1,1}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right)\hat{y} + \frac{1}{i}\sqrt{\frac{2\pi}{3}}\left(Y_{1,1}(\theta,\phi) + Y_{1,-1}(\theta,\phi)\right)\hat{x}\right] = \frac{\mu_0 \sigma \omega a}{3}\frac{r_<}{r_>^2}\left[\sin\theta\cos\phi\,\hat{y} - \sin\theta\sin\phi\,\hat{x}\right]$$

$$\Rightarrow \boxed{\vec{A}(\vec{x}) = \frac{\mu_0 \sigma \omega a}{3}\frac{r_<}{r_>^2}\sin\theta\,\hat{\phi}} \qquad \left(\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right)$$

For $r<a$, we have:

$$\vec{A}(\vec{x}) = \frac{\mu_0 \sigma \omega}{a} r\sin\theta\,\hat{\phi} \Rightarrow \boxed{\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x}) = \frac{2\mu_0 \sigma \omega}{3a}\hat{z}}$$

For $r>a$, we have:

$$\vec{A}(\vec{x}) = \frac{\mu_0 \sigma \omega a^2}{r^2}\sin\theta\,\hat{\phi} \Rightarrow \boxed{\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x}) = \frac{\mu_0 \sigma \omega a^2}{3r^2}\sin\theta\,\hat{\phi}}$$